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A note on the existence and uniqueness of solutions of frequency domain elastic wave problems: *A priori* estimates in $\boldsymbol{H}^1$ ☆

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ABSTRACT

In this note, we provide existence and uniqueness results for frequency domain elastic wave problems. These problems are posed on the complement of a bounded domain $\Omega \subset \mathbb{R}^3$ (the scatterer). The boundary condition at infinity is given by the Kupradze–Sommerfeld radiation condition and involves different Sommerfeld conditions on different components of the field. Our results are obtained by setting up the problem as a variational problem in the Sobolev space $\boldsymbol{H}^1$ on a bounded domain. We use a nonlocal boundary condition which is related to the Dirichlet to Neumann conditions used for acoustic and electromagnetic scattering problems. We obtain stability results for the source problem, a necessary ingredient for the analysis of numerical methods for this problem based on finite elements or finite differences.

1. Introduction

The goal of this paper is to provide new existence, uniqueness and stability results for the solutions of frequency domain elastic wave scattering problems in the natural Sobolev spaces. Such estimates are necessary for the analysis of numerical methods based on finite elements. These problems are posed on the complement of a bounded domain (the scatterer) and involve pressure and shear waves with different wave numbers. The far field radiation condition is the so-called "Kupradze–Sommerfeld" condition which prescribes two different Sommerfeld conditions on the two types of waves.

The existence and uniqueness of solutions for the elastic wave scattering problem has been investigated using integral equation techniques [7–9]. These papers provide classical solutions for domains with suitably smooth boundaries (e.g., C^2) and problems with suitably smooth boundary data.

In this paper, we shall formulate the elastic wave problem as a variational problem on a bounded subdomain Ω_R (the exterior of the scatterer intersected with a ball of radius R containing the scatterer) with a non-local boundary condition provided by a Dirichlet to Neumann map. The non-local boundary condition builds in the Kupradze–Sommerfeld radiation condition. We shall show that this variational problem is well posed on $\boldsymbol{H}^1(\Omega_R)$, i.e., the solution of the elastic wave problem is in $\boldsymbol{H}^1(\Omega_R)$ and satisfies appropriate *a priori* inequalities. These results hold for a scatterer with only a Lipschitz continuous boundary and boundary data in an appropriate Sobolev space. The solution which we obtain is independent of Ω_R in the sense that if Ω_R and Ω_{R_1} are two such domains then their solutions coincide on $\Omega_R \cap \Omega_{R_1}$. Although the Dirichlet to Neumann approach is natural and has been successfully employed for the analysis of acoustic and Maxwell problems [10], it has yet to be extended to the elastic wave problem.

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Without loss of generality, we may take Ω_R to have a spherical outer boundary. It is then possible to write the solution of the elastic wave scattering problem outside of Ω_R in terms of a series of vector wave functions and gradients of functions which satisfy the Helmholtz equation (see Section 2). For acoustic and electromagnetic problems, the Dirichlet to Neumann map can be defined directly from the series. However, it seems that for the elastic wave problem, the Dirichlet to Neumann map on the outer boundary of Ω_R can only be formally expanded in terms of this series (this is what is proposed in [6]). Our approach is somewhat different. We use the series to extend functions outside of Ω_R and show that the extended function is locally in $\boldsymbol{H}^1$. This allows us to define the Dirichlet to Neumann map by differentiating the resulting extension.

As we shall see, there are several equivalent variational formulations for the elastic wave problem. It will be convenient to use one such formulation to conclude uniqueness and another to verify an inf–sup condition which leads to existence.

Our results are important from the computational point of view. Indeed, stability in $\boldsymbol{H}^1$ on a bounded domain is a necessary ingredient for the analysis of any discrete approximation based on finite elements or finite differences. The Kupradze–Sommerfeld radiation condition, though, provides additional numerical modeling difficulties. In a subsequent paper [3], we shall use the existence and uniqueness results of this paper as one step in the analysis of numerical approximations based on the “so-called” perfectly matched layer (PML). PML represents an efficient way to develop approximate boundary conditions for this problem and avoids the computational splitting of the solution.

The outline of the remainder of the paper is as follows. In Section 2, we formulate the elastic wave scattering problem and the Kupradze–Sommerfeld outgoing radiation condition. In Section 3, we set up the variational formulation using Dirichlet to Neumann maps and show existence and uniqueness of the solution (locally, in $\boldsymbol{H}^1$).

2. Formulation of the elastic wave problem

In this section, we formulate the elastic wave problem and its far field boundary conditions. Let Ω be a bounded domain containing the origin and Ω^c denote its complement. We seek a vector valued function $\boldsymbol{u} \in \boldsymbol{H}^1_{\mathrm{loc}}(\Omega^c) \equiv (H^1_{\mathrm{loc}}(\Omega^c))^3$ satisfying (the weak equation)

$$k^2\boldsymbol{u} + \boldsymbol{\Delta u} + \gamma \nabla\nabla \cdot \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Omega^c \tag{2.1}$$

and

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma \equiv \partial\Omega. \tag{2.2}$$

Here γ and k are positive real numbers and $\boldsymbol{g}$ is given in $\boldsymbol{H}^{1/2}(\partial\Omega)$.

To complete the problem definition, we need to pose boundary conditions at infinity corresponding to outgoing waves. Let B_R be a ball of radius R containing Ω. We first note that any function $\boldsymbol{u}$ satisfying (2.1) is smooth away from Γ since it satisfies a constant coefficient elliptic equation. We set

$$\psi \equiv k_1^{-2} \nabla \cdot \boldsymbol{u}. \tag{2.3}$$

Then

$$k^2\psi + (1+\gamma)\Delta\psi = 0 \quad \text{in } B_R^c. \tag{2.4}$$

Here $k_1 = k/\sqrt{1+\gamma}$. We require $\boldsymbol{u}$ to be such that ψ satisfies the Sommerfeld radiation condition, i.e.,

$$\lim_{r\to\infty} r\left(\frac{\partial\psi}{\partial r} - ik_1\psi\right) = 0. \tag{2.5}$$

Define

$$\boldsymbol{\zeta} = \boldsymbol{u} - \nabla\psi. \tag{2.6}$$

By construction, $\Delta\psi = -k_1^2\psi = \nabla \cdot \boldsymbol{u}$ so $\nabla \cdot \boldsymbol{\zeta} = 0$. Moreover, each component of $\nabla\psi$ satisfies (2.4) which implies that $\nabla\psi$ satisfies (2.1). It follows that $\boldsymbol{\zeta}$ satisfies

$$\boldsymbol{0} = k^2\boldsymbol{\zeta} + \boldsymbol{\Delta\zeta} = k^2\boldsymbol{\zeta} - \nabla\times\nabla\times\boldsymbol{\zeta} \quad \text{in } B_R^c. \tag{2.7}$$

We require $\boldsymbol{\zeta}$ to satisfy the Silver–Müller radiation condition, i.e.,

$$\lim_{r\to\infty} r(\nabla\times\boldsymbol{\zeta} \times \hat{\boldsymbol{x}} - ik\boldsymbol{\zeta}) = \boldsymbol{0}. \tag{2.8}$$

Remark 2.1. We shall see below that a decomposition of $\boldsymbol{u}$ satisfying (2.4)–(2.8) is uniquely determined by the trace of $\boldsymbol{u}$ on the boundary of B_R.

Remark 2.2. We note that by (2.6), we have constructed a decomposition of $\boldsymbol{u}$ of the form

$$\boldsymbol{u} = \boldsymbol{\zeta} + \boldsymbol{\psi}$$

with $\boldsymbol{\zeta}$ solenoidal and $\boldsymbol{\psi}$ irrotational. It follows from (2.7) and (2.8) that $\boldsymbol{\zeta}$ also satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial\boldsymbol{\zeta}}{\partial n} - ik\boldsymbol{\zeta}\right) = \mathbf{0}.$$

In addition, it can be shown that $\boldsymbol{\psi} \equiv \nabla\psi$ satisfies

$$\lim_{r\to\infty} r\left(\frac{\partial\boldsymbol{\psi}}{\partial n} - ik_1\boldsymbol{\psi}\right) = \mathbf{0}.$$

These are the classical Kupradze–Sommerfeld radiation conditions.

Conversely, if we have a decomposition satisfying the classical Kupradze–Sommerfeld conditions then there is a potential $\tilde{\psi}$ with $\nabla\tilde{\psi} = \boldsymbol{\psi}$ outside of B_R. This potential, modified by a suitable constant, along with $\boldsymbol{\zeta}$ gives rise to a decomposition of $\boldsymbol{u}$ satisfying (2.4)–(2.8). By Remark 2.1, the resulting decomposition is the same as ours, i.e,. $\tilde{\psi} = \psi + c$ where ψ is defined by (2.3). Thus, the boundary conditions defined using (2.3)–(2.8) are the same as the classical Kupradze–Sommerfeld conditions. For the purpose of our subsequent analysis, it is more convenient to work with ψ (instead of $\boldsymbol{\psi}$).

Now, ψ can be expanded outside of B_R in a series of the form

$$\psi(\boldsymbol{x}) = \sum_{n=0}^{\infty}\sum_{|m|\leqslant n} \gamma_{n,m} p_n(r) Y_{n,m}(\hat{\boldsymbol{x}}). \tag{2.9}$$

Here $p_n(r) \equiv h_n^{(1)}(k_1 r)$, $h_n^{(1)}$ is the Hankel function of the first kind of order n, $Y_{n,m}$ are spherical harmonics, $r = |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$.

Let $q_n(r) = h_n^{(1)}(kr)$, $\boldsymbol{V}_{n,m} = \hat{\boldsymbol{x}} \times \boldsymbol{U}_{n,m}$ where

$$\boldsymbol{U}_{n,m} = \boldsymbol{U}_{n,m}(\theta,\phi) = \frac{1}{\sqrt{\lambda_n}}\left[\frac{\partial Y_{n,m}}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{\sin(\theta)}\frac{\partial Y_{n,m}}{\partial\phi}\hat{\boldsymbol{\phi}}\right].$$

Here $\lambda_n = n(n+1)$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$ are the spherical unit vectors while ϕ and θ are the corresponding spherical coordinates. Theorem 9.17 [10] implies that $\boldsymbol{\zeta}$ can be expanded

$$\boldsymbol{\zeta} = \sum_{n=1}^{\infty}\sum_{|m|\leqslant n}\left[\alpha_{n,m} q_n(r)\boldsymbol{V}_{n,m} + \beta_{n,m}\nabla\times\left(q_n(r)\boldsymbol{V}_{n,m}\right)\right]. \tag{2.10}$$

We shall see below that the coefficients, $\alpha_{n,m}$, $\beta_{n,m}$, and $\gamma_{n,m}$ are well defined from the trace of $\boldsymbol{u}$ on ∂B_R as long as it is in $\boldsymbol{L}^2(\partial B_R)$.

Thus, we seek a vector function $\boldsymbol{u}$ which satisfies (2.1), (2.2) and has an expansion outside of B_R of the form

$$\boldsymbol{u} = \boldsymbol{\zeta} + \nabla\psi = \sum_{n=0}^{\infty}\sum_{|m|\leqslant n}\left[\alpha_{n,m} q_n(r)\boldsymbol{V}_{n,m} + \beta_{n,m}\nabla\times\left(q_n(r)\boldsymbol{V}_{n,m}\right) + \gamma_{n,m}\nabla\left(p_n(r)Y_{n,m}\right)\right]. \tag{2.11}$$

Here we have set $\alpha_{0,0} = \beta_{0,0} = 0$ and $\boldsymbol{V}_{0,0} = \boldsymbol{U}_{0,0} = \mathbf{0}$ for convenience of notation. Any component of this series satisfies (2.5) and (2.8) hence so will ψ and $\boldsymbol{\zeta}$ provided that the coefficients have sufficient decay as m and n become large (as we demonstrate below).

3. Existence and uniqueness for the elastic wave problem

In this section, we prove existence, uniqueness and some regularity results for the time-harmonic elastic wave problem. As this is done in an $\boldsymbol{H}^1$ setting, we obtain solutions which are in $(H^1(D))^3$ for any bounded subset D of Ω^c.

The series (2.11) will play a central role in our analysis. For any $\boldsymbol{u}$ given by (2.11) we write

$$\boldsymbol{u} = \sum_{n=0}^{\infty}\sum_{|m|\leqslant n}\boldsymbol{u}_{n,m}$$

where $\boldsymbol{u}_{n,m}$ is defined by the right-hand side of (2.11). As observed in [6], outside of B_R,

$$\boldsymbol{u}_{n,m} = \alpha_{n,m} q_n\boldsymbol{V}_{n,m} + \left(\frac{\gamma_{n,m}\sqrt{\lambda_n}p_n}{r} - \frac{\beta_{n,m}}{r}(rq_n)'\right)\boldsymbol{U}_{n,m} + \left(\gamma_{n,m}p_n' - \frac{\beta_{n,m}\sqrt{\lambda_n}q_n}{r}\right)Y_{n,m}\hat{\boldsymbol{x}}. \tag{3.1}$$

This follows from the identities

$$\nabla\times(q_n \boldsymbol{V}_{n,m}) = -\frac{\sqrt{\lambda_n}q_n}{r} Y_{n,m}\hat{\boldsymbol{x}} - \frac{1}{r}(rq_n)'\boldsymbol{U}_{n,m}$$

and

$$\nabla(p_n Y_{n,m}) = p_n' Y_{n,m}\hat{\boldsymbol{x}} + \frac{\sqrt{\lambda_n}p_n}{r}\boldsymbol{U}_{n,m}.$$

The components $\boldsymbol{V}_{n,m}$, $\boldsymbol{U}_{n,m}$ and $Y_{n,m}\hat{\boldsymbol{x}}$ form an orthonormal basis (when n and m are varied) for $\boldsymbol{L}^2(S_1)$ where S_1 denotes the unit sphere. Indeed, $\{\boldsymbol{V}_{n,m}, \boldsymbol{U}_{n,m}\}$ form an orthonormal basis for the tangential fields in $\boldsymbol{L}^2(S_1)$ (cf. [10]) and $\{Y_{n,m}\hat{\boldsymbol{x}}\}$ gives an orthonormal basis for the radial fields in $\boldsymbol{L}^2(S_1)$. This means that any function $\boldsymbol{w}$ defined on $\Gamma_B \equiv \partial B_R$ can be expanded

$$\boldsymbol{w} = \sum_{n=0}^{\infty}\sum_{|m|\leqslant n}(a_{n,m}\boldsymbol{V}_{n,m} + b_{n,m}\boldsymbol{U}_{n,m} + c_{n,m}Y_{n,m}\hat{\boldsymbol{x}}). \tag{3.2}$$

Lemma 3.1. *Given coefficients $b_{n,m}$ and $c_{n,m}$, there is a unique pair $\beta_{n,m}$, $\gamma_{n,m}$ satisfying (compare with (3.1))*

$$\left(\frac{\gamma_{n,m}\sqrt{\lambda_n}p_n}{R} - \frac{\beta_{n,m}}{R}(Rq_n)'\right)\boldsymbol{U}_{n,m} + \left(\gamma_{n,m}p_n' - \frac{\beta_{n,m}\sqrt{\lambda_n}q_n}{R}\right)Y_{n,m}\hat{\boldsymbol{x}} = b_{n,m}q_n\boldsymbol{U}_{n,m} + c_{n,m}p_nY_{n,m}\hat{\boldsymbol{x}}. \tag{3.3}$$

Here p_n, p_n', q_n, q_n' are all evaluated at R. Moreover, there is a positive constant $C = C(R)$ independent of n satisfying

$$|\gamma_{n,m}p_n|^2 + |\beta_{n,m}q_n|^2 \leqslant C(1+n)^2\big(|b_{n,m}q_n|^2 + |c_{n,m}p_n|^2\big).$$

Proof. The above system is

$$\begin{pmatrix} -\frac{q_n'}{q_n} - \frac{1}{R} & \frac{\sqrt{\lambda_n}}{R} \\ -\frac{\sqrt{\lambda_n}}{R} & \frac{p_n'}{p_n}\end{pmatrix}\begin{pmatrix}\beta_{n,m}q_n \\ \gamma_{n,m}p_n\end{pmatrix} = \begin{pmatrix} b_{n,m}q_n \\ c_{n,m}p_n\end{pmatrix}. \tag{3.4}$$

Its determinant is

$$\mathrm{Det} = \frac{n(n+1)}{R^2} - \left(\frac{p_n'}{p_n}\right)\left(\frac{q_n'}{q_n} + \frac{1}{R}\right). \tag{3.5}$$

We first observe that Det does not vanish for any n. Indeed, the imaginary part of p_n'/p_n is

$$-i\frac{1}{2|p_n|^2}\big(p_n'\bar{p}_n - \bar{p}_n' p_n\big) = \frac{ik_1}{2|p_n|^2}W\big(h_n^{(1)}(k_1R), h_n^{(2)}(k_1R)\big) = \frac{1}{k_1|p_n|^2R^2}$$

where we used the well-known Wronskian identity $W(h_n^{(1)}(r), h_n^{(2)}(r)) = -2i/r^2$. An identical argument shows that the imaginary part of $q_n'/q_n + 1/R$ is $1/(k|q_n|^2R^2)$. It follows that the product of these two terms cannot be real and positive, i.e. $\mathrm{Det} \neq 0$.

We next develop an asymptotic bound for the determinant valid for large n. Using the identity

$$\big(h_n^{(1)}\big)'(z) = \frac{n}{z}h_n^{(1)}(z) - h_{n+1}^{(1)}(z)$$

gives

$$\frac{p_n'}{p_n} = \frac{1}{R}\left(n - k_1R\frac{p_{n+1}}{p_n}\right). \tag{3.6}$$

Using the identity

$$h_{n-1}^{(1)}(r) + h_{n+1}^{(1)}(r) = \frac{2n+1}{r}h_n^{(1)}(r) \tag{3.7}$$

and the asymptotic relation

$$h_n^{(1)}(z) = \frac{(2n-1)!!}{iz^{n+1}}\big(1 + O(1/n)\big), \tag{3.8}$$

where $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$, we obtain

$$k_1R\frac{p_{n+1}}{p_n} = 2n + 1 - \frac{k_1^2R^2}{(2n-1)} + O\big(1/n^2\big).$$

Putting this in (3.6) gives

$$\frac{p_n'}{p_n} = \frac{1}{R}\left(-n-1+\frac{k_1^2R^2}{2n-1}+O\left(1/n^2\right)\right).$$

Inserting this and the analogous expression for q_n'/q_n into (3.5) yields

$$\mathrm{Det} = \frac{1}{2}\left(k_1^2+k^2\right)+O(1/n).$$

We conclude that there is a constant C not depending on n such

$$|\mathrm{Det}|^{-1} \leqslant C$$

for all n.

We will show that the absolute value of each entry appearing in the two by two matrix in (3.4) can be bounded by $C(n+1)$. The lemma will then follow from Cramer's rule.

From (3.6) and (3.7) we have, for $n \geqslant 1$,

$$\frac{p_n'}{p_n} = \frac{1}{R}\left(-n-1+k_1R\frac{p_{n-1}}{p_n}\right). \tag{3.9}$$

Finally we will show that $|p_{n-1}/p_n|$ is uniformly bounded and hence, with (3.9), gives

$$\left|\frac{p_n'}{p_n}\right| \leqslant C(n+1). \tag{3.10}$$

We will, in fact, prove that $|p_{n-1}/p_n| \leqslant 1$. This is the same as

$$\left|h_n^{(1)}(r)\right|^2 \geqslant \left|h_{n-1}^{(1)}(r)\right|^2. \tag{3.11}$$

The following expression for $|h_n^{(1)}(r)|^2$ may be found in [1]:

$$\left|h_n^{(1)}(r)\right|^2 = \frac{1}{r^2}\sum_{k=0}^{n}\frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2r)^{2k-2n}. \tag{3.12}$$

We drop the first term and change the summation index to obtain

$$\left|h_n^{(1)}(r)\right|^2 \geqslant \frac{1}{r^2}\sum_{k=1}^{n}\frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2r)^{2k-2n} = \frac{1}{r^2}\sum_{k=0}^{n-1}\frac{(2n-k-1)}{(k+1)}\frac{(2(n-1)-k)!(2(n-1)-2k)!}{k![(n-1-k)!]^2}(2r)^{2k-2(n-1)}$$

from which (3.11) follows. The analogous bound holds for $|\frac{q_n'}{q_n}|$. This completes the proof of the lemma. □

The above lemma shows that any function $\boldsymbol{w}$ in $\boldsymbol{L}^2(\Gamma_R)$ can be expanded in a series of the form

$$\boldsymbol{w}(R,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{|m|\leqslant n}\left[\alpha_{n,m}q_n(R)\boldsymbol{V}_{n,m}+\beta_{n,m}\nabla\times\left(q_n(R)\boldsymbol{V}_{n,m}\right)+\gamma_{n,m}\nabla\left(p_n(R)Y_{n,m}\right)\right] \tag{3.13}$$

by first expanding $\boldsymbol{w}$ as in (3.2) and applying the lemma. Accordingly, $\boldsymbol{w}$ can be extended outside of Ω_R by

$$\tilde{\boldsymbol{w}}(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{|m|\leqslant n}\left[\alpha_{n,m}q_n(r)\boldsymbol{V}_{n,m}+\beta_{n,m}\nabla\times\left(q_n(r)\boldsymbol{V}_{n,m}\right)+\gamma_{n,m}\nabla\left(p_n(r)Y_{n,m}\right)\right]. \tag{3.14}$$

Before proceeding, we characterize the boundary Sobolev norms in terms of the series. Let Δ_1 denote the surface Laplacian. Then,

$$\Delta_1\boldsymbol{V}_{n,m} = \lambda_n\boldsymbol{V}_{n,m}, \qquad \Delta_1\boldsymbol{U}_{n,m} = \lambda_n\boldsymbol{U}_{n,m}, \quad \text{and} \quad \Delta_1(Y_{n,m}\hat{\boldsymbol{x}}) = \lambda_nY_{n,m}\hat{\boldsymbol{x}}.$$

Accordingly, the boundary Sobolev norms are given in terms of the coefficients, i.e., if $\boldsymbol{w}$ is expanded as in (3.2) then $\boldsymbol{w} \in \boldsymbol{H}^s(\Gamma_R)$ if and only if the series

$$\sum_{n=0}^{\infty}\sum_{|m|\leqslant n}(1+\lambda_n)^s\left(|a_{n,m}|^2+|b_{n,m}|^2+|c_{n,m}|^2\right) \tag{3.15}$$

converges.

Our approach to analyze problem (2.1) with boundary conditions (2.2), (2.5), (2.8) is to pose the problem in $\boldsymbol{H}^1(\Omega_R)$ with an outer boundary condition provided by an appropriate Dirichlet to Neumann map ("*DN*") on Γ_R. Let $\widetilde{\boldsymbol{H}}^1_0(\Omega_R)$ denote the functions in $\boldsymbol{H}^1(\Omega_R)$ which vanish on $\partial\Omega$. We define the form for $\boldsymbol{w}, \boldsymbol{v} \in \widetilde{\boldsymbol{H}}^1_0(\Omega_R)$, by

$$A(\boldsymbol{w}, \boldsymbol{v}) = k^2(\boldsymbol{w}, \boldsymbol{v})_{\Omega_R} - (\nabla \boldsymbol{w}, \nabla \boldsymbol{v})_{\Omega_R} - \gamma(\nabla \cdot \boldsymbol{w}, \nabla \cdot \boldsymbol{v})_{\Omega_R} + \big(DN(\tilde{\boldsymbol{w}}), \boldsymbol{v}\big)_{\Gamma_R}. \tag{3.16}$$

The boundary term is

$$\big(DN(\tilde{\boldsymbol{w}}), \boldsymbol{v}\big)_{\Gamma_R} = \left(\frac{\partial \tilde{\boldsymbol{w}}}{\partial \hat{\boldsymbol{x}}}, \boldsymbol{v}\right)_{\Gamma_R} + \gamma(\nabla \cdot \tilde{\boldsymbol{w}}, \boldsymbol{v} \cdot \hat{\boldsymbol{x}})_{\Gamma_R}. \tag{3.17}$$

The series defining $\tilde{\boldsymbol{w}}$ and all of its derivatives converge uniformly away from Γ_R (cf., [5,10]). We shall subsequently show that resulting limit coincides with a function in $\boldsymbol{H}^1(B_{2R} \setminus B_R)$ for which the terms appearing on the right-hand side of (3.17) make sense.

Remark 3.1. It is not clear for an arbitrary $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega_R)$ whether the derivatives appearing on the right-hand side of (3.17) can be computed by term by term differentiation (at Γ_R) of the series defining $\tilde{\boldsymbol{w}}$.

Our analysis and the definition of the Dirichlet to Neumann operator involves going outside of Ω_R. We assume without loss that (the weak form of) the problem

$$\begin{aligned} &k^2\boldsymbol{v} + \boldsymbol{\Delta}\boldsymbol{v} + \gamma\nabla\nabla\cdot\boldsymbol{v} = \boldsymbol{f} \quad \text{in } \Omega_{2R} \setminus \Omega_R, \\ &\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial(\Omega_{2R} \setminus \Omega_R) \end{aligned} \tag{3.18}$$

is well posed in $\boldsymbol{H}^1_0(\Omega_{2R} \setminus \Omega_R)$. If necessary, one can change R slightly to guarantee stability. Indeed, the above operator and boundary conditions without lower-order term on $\Omega_{2R} \setminus \Omega_R$ has eigenvalues that are isolated with only accumulation at infinity. The change of variables $R \to R + \epsilon$ results in an eigenvalue shift $\lambda \to \lambda(1 + \epsilon/R)^{-2}$.

Given $\boldsymbol{w} \in \widetilde{\boldsymbol{H}}^1_0(\Omega_R)$, we let $\tilde{\boldsymbol{w}}$ denote the extended function, i.e.,

$$\tilde{\boldsymbol{w}}(\boldsymbol{x}) = \begin{cases} \boldsymbol{w}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \overline{\Omega}_R, \\ \tilde{\boldsymbol{w}}(\boldsymbol{x}) \text{ given by (3.14)} & \text{for } |\boldsymbol{x}| > R. \end{cases}$$

The two series (2.9) and (2.10) and their derivatives converge uniformly on compact sets bounded away from Ω_R. It follows that $\tilde{\boldsymbol{w}}$ is smooth outside of Ω_R and satisfies

$$k^2\tilde{\boldsymbol{w}} + \boldsymbol{\Delta}\tilde{\boldsymbol{w}} + \gamma\nabla\nabla\cdot\tilde{\boldsymbol{w}} = \boldsymbol{0} \quad \text{in } \Omega_{2R} \setminus \Omega_R. \tag{3.19}$$

Note that we also know that $\tilde{\boldsymbol{w}}$ is in $\boldsymbol{H}^{1/2}(\Gamma_R)$ since $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega_R)$ implies that $\boldsymbol{w} \in \boldsymbol{H}^{1/2}(\Gamma_R)$ and $\boldsymbol{w} = \tilde{\boldsymbol{w}}$ on Γ_R. The stability of (3.18) implies that $\tilde{\boldsymbol{w}}$ is in $\boldsymbol{H}^1(\Omega_{2R} \setminus \Omega_R)$ and satisfies

$$\|\tilde{\boldsymbol{w}}\|_{1,\Omega_{2R}\setminus\Omega_R} \leqslant C\big(\|\boldsymbol{w}\|_{1/2,\Gamma_R} + \|\tilde{\boldsymbol{w}}\|_{1/2,\Gamma_{2R}}\big). \tag{3.20}$$

We will estimate the norm on Γ_{2R} now.

We have already proved that

$$\left|\frac{q_n'}{q_n}\right| \leqslant C(n+1)$$

and hence

$$\left|\left(\frac{\gamma_{n,m}\sqrt{\lambda_n}\,p_n(2R)}{2R} - \frac{\beta_{n,m}}{2R}\big(rq_n(r)\big)'\big|_{r=2R}\right)\right| \leqslant C(n+1)\big(|\gamma_{n,m}|\big|p_n(2R)\big| + |\beta_{n,m}|\big|q_n(2R)\big|\big).$$

The left-hand side above is the absolute value of the coefficient of $\boldsymbol{U}_{n,m}$ in the orthogonal expansion for $\tilde{\boldsymbol{w}}$ on Γ_R (similar to (3.1)). Now the relation (3.8) holds uniformly in n and z as long z varies in $[k_1R, 2k_1R]$. Thus,

$$\big|p_n(2R)\big| \leqslant C2^{-n}\big|p_n(R)\big|$$

with a similar estimate for $|q_n(2R)|$. Thus,

$$\left|\left(\frac{\gamma_{n,m}\sqrt{\lambda_n}\,p_n(2R)}{2R} - \frac{\beta_{n,m}}{2R}\big(rq_n(r)\big)'\big|_{r=2R}\right)\right| \leqslant C(n+1)2^{-n}\big(|\gamma_{n,m}|\big|p_n(R)\big| + |\beta_{n,m}|\big|q_n(R)\big|\big).$$

A similar estimate holds for the $Y_{n,m}\hat{\boldsymbol{x}}$ coefficient while the remaining coefficient satisfies

$$\big|\alpha_{n,m}q_n(2R)\big| \leqslant C2^{-n}|\alpha_{n,m}|\big|q_n(R)\big|.$$

It follows by (3.15) that

$$\begin{aligned}\|\tilde{\boldsymbol{w}}\|^2_{1/2,\Gamma_{2R}} &\leqslant C\sum_{n=0}^{\infty}\sum_{|m|\leqslant n}(1+n)2^{-2n}\big[|\alpha_{n,m}|^2\big|q_n(R)\big|^2+(n+1)^2\big(|\gamma_{n,m}|^2\big|p_n(R)\big|^2+|\beta_{n,m}|^2\big|q_n(R)\big|^2\big)\big]\\ &\leqslant C\sum_{n=0}^{\infty}\sum_{|m|\leqslant n}(1+n)2^{-2n}\big[|\alpha_{n,m}|^2\big|q_n(R)\big|^2+(n+1)^4\big(|c_{n,m}|^2\big|p_n(R)\big|^2+|b_{n,m}|^2\big|q_n(R)\big|^2\big)\big] \qquad (3.21)\end{aligned}$$

where $b_{n,m}$ and $c_{n,m}$ satisfy (3.3). Applying (3.15) and the geometric decay of 2^{-2n} shows that

$$\|\tilde{\boldsymbol{w}}\|_{1/2,\Gamma_{2R}}\leqslant C\|\boldsymbol{w}\|_{0,\Gamma_R}. \qquad (3.22)$$

Remark 3.2. The above argument can be used to show that the map $\boldsymbol{w}\to\tilde{\boldsymbol{w}}$ is compact from $\boldsymbol{H}^1(\Omega_R)$ into $\boldsymbol{H}^{1/2}(\Gamma_{2R})$.

The terms on the right-hand side of (3.17) now make sense since the extended function $\tilde{\boldsymbol{w}}$ is in $\boldsymbol{H}^1(\Omega_{2R})$ when $\boldsymbol{w}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)$. Hence the form given by (3.16) makes sense. We also define the form, for $\boldsymbol{w},\boldsymbol{v}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)$,

$$A_1(\boldsymbol{w},\boldsymbol{v})=k^2(\tilde{\boldsymbol{w}},\tilde{\boldsymbol{v}})_{\Omega_{2R}}-(\nabla\tilde{\boldsymbol{w}},\nabla\tilde{\boldsymbol{v}})_{\Omega_{2R}}-\gamma(\nabla\cdot\tilde{\boldsymbol{w}},\nabla\cdot\tilde{\boldsymbol{v}})_{\Omega_{2R}}+\big(DN_1(\tilde{\boldsymbol{w}}),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}. \qquad (3.23)$$

Here DN_1 denotes the Dirichlet to Neumann operator on Γ_{2R} given by

$$\big(DN_1(\tilde{\boldsymbol{w}}),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}=\left(\frac{\partial\tilde{\boldsymbol{w}}}{\partial\hat{\boldsymbol{x}}},\tilde{\boldsymbol{v}}\right)_{\Gamma_{2R}}+\gamma(\nabla\cdot\tilde{\boldsymbol{w}},\tilde{\boldsymbol{v}}\cdot\hat{\boldsymbol{x}})_{\Gamma_{2R}}.$$

There is no problem in the definition above as $\tilde{\boldsymbol{w}}$ is smooth near Γ_{2R}. Moreover, since the series and all of its derivatives converge uniformly near Γ_{2R}, DN_1 can also be expressed by term by term differentiation of the series.

The form A_1 still can be thought of as a standard (local) bilinear form on Ω_R plus a nonlocal boundary term. The local part consists of the volume integrals restricted to Ω_R while the nonlocal boundary term involves the volume integrals restricted to $\Omega_{2R}\setminus\Omega_R$ and the DN_2 term.

Similarly we define the form

$$A_2(\boldsymbol{w},\boldsymbol{v})=k^2(\tilde{\boldsymbol{w}},\tilde{\boldsymbol{v}})_{\Omega_{2R}}-(\nabla\times\tilde{\boldsymbol{w}},\nabla\times\tilde{\boldsymbol{v}})_{\Omega_{2R}}-(1+\gamma)(\nabla\cdot\tilde{\boldsymbol{w}},\nabla\cdot\tilde{\boldsymbol{v}})_{\Omega_{2R}}+\big(DN_2(\tilde{\boldsymbol{w}}),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}. \qquad (3.24)$$

Here DN_2 denotes the Dirichlet to Neumann operator on Γ_{2R} given by

$$\big(DN_2(\tilde{\boldsymbol{w}}),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}=-(\hat{\boldsymbol{x}}\times\nabla\times\tilde{\boldsymbol{w}},\tilde{\boldsymbol{w}})_{\Gamma_{2R}}+(1+\gamma)(\nabla\cdot\tilde{\boldsymbol{w}},\tilde{\boldsymbol{w}}\cdot\hat{\boldsymbol{x}})_{\Gamma_{2R}}.$$

Proposition 1. *The forms $A(\cdot,\cdot)$, $A_1(\cdot,\cdot)$ and $A_2(\cdot,\cdot)$ are all bounded on $\widetilde{\boldsymbol{H}}^1_0(\Omega_R)\times\widetilde{\boldsymbol{H}}^1_0(\Omega_R)$ and coincide.*

Proof. Let $\boldsymbol{w}$ and $\boldsymbol{v}$ be smooth and vanish on Γ. We clearly have

$$\big(k^2\tilde{\boldsymbol{w}}+\boldsymbol{\Delta}\tilde{\boldsymbol{w}}+\gamma\nabla\nabla\cdot\tilde{\boldsymbol{w}},\tilde{\boldsymbol{v}}\big)_{\Omega_{2R}\setminus\Omega_R}=0.$$

Applying integration by parts to the above identity and adding it to $A(\boldsymbol{w},\boldsymbol{v})$ shows that $A(\boldsymbol{w},\boldsymbol{v})=A_1(\boldsymbol{w},\boldsymbol{v})$.

Next, let χ be a smooth cutoff function which is one in Ω_R and near $\partial\Gamma_R$ while vanishing near Γ_{2R}. We decompose $\tilde{\boldsymbol{w}}=\chi\tilde{\boldsymbol{w}}+(1-\chi)\tilde{\boldsymbol{w}}$. Now let $\boldsymbol{\phi}_n$ be smooth, have support in Ω_{2R} and be such that $\boldsymbol{\phi}_n$ converges to $\chi\tilde{\boldsymbol{w}}$ in $H^1(\Omega_{2R})$ as $n\to\infty$. Set $\boldsymbol{\psi}_n=\boldsymbol{\phi}_n+(1-\chi)\tilde{\boldsymbol{w}}$. Then,

$$\begin{aligned}&k^2(\boldsymbol{\psi}_n,\tilde{\boldsymbol{v}})_{\Omega_{2R}}-(\nabla\boldsymbol{\psi}_n,\nabla\tilde{\boldsymbol{v}})_{\Omega_{2R}}-\gamma(\nabla\cdot\boldsymbol{\psi}_n,\nabla\cdot\tilde{\boldsymbol{v}})_{\Omega_{2R}}+\big(DN_1(\boldsymbol{\psi}_n),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}\\ &\quad=\big(k^2\boldsymbol{\psi}_n+\boldsymbol{\Delta}\boldsymbol{\psi}_n+\gamma\nabla\nabla\cdot\boldsymbol{\psi}_n,\tilde{\boldsymbol{v}}\big)_{\Omega_{2R}}\\ &\quad=\big(k^2\boldsymbol{\psi}_n-\nabla\times\nabla\times\boldsymbol{\psi}_n+(1+\gamma)\nabla\nabla\cdot\boldsymbol{\psi}_n,\tilde{\boldsymbol{v}}\big)_{\Omega_{2R}}\\ &\quad=k^2(\boldsymbol{\psi}_n,\tilde{\boldsymbol{v}})_{\Omega_{2R}}-(\nabla\times\boldsymbol{\psi}_n,\nabla\times\tilde{\boldsymbol{v}})_{\Omega_{2R}}-(1+\gamma)(\nabla\cdot\boldsymbol{\psi}_n,\nabla\cdot\tilde{\boldsymbol{v}})_{\Omega_{2R}}+\big(DN_2(\boldsymbol{\psi}_n),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}.\end{aligned}$$

Taking the limit as $n\to\infty$ shows that $A_1(\boldsymbol{w},\boldsymbol{v})=A_2(\boldsymbol{w},\boldsymbol{v})$.

Since p_n and q_n satisfy second-order differential equations (similar to those satisfied by the Hankel functions), the derivatives appearing in DN_1 can be bounded by the coefficients of (3.1), e.g.,

$$\left\|\left[\left(\frac{\gamma_{n,m}\sqrt{\lambda_n}p_n}{r}-\frac{\beta_{n,m}}{r}(rq_n)'\right)'\boldsymbol{U}_{n,m}\right]\right\|\leqslant C(n+1)^2\big[|\gamma_{n,m}||p_n|+|\beta_{n,m}||q_n|\big].$$

The arguments leading to (3.22) can be used to show that $DN_1(\tilde{\boldsymbol{w}})$ is a compact map of $\boldsymbol{H}^{1/2}(\Gamma_R)$ into $\boldsymbol{H}^{-1/2}(\Gamma_{2R})$. Thus, it follows from (3.20) and (3.22) that A_1 is bounded. That A, A_1 and A_2 coincide follows from a density argument. □

Remark 3.3. The above proposition shows that we can simultaneously change the form in the domain and Dirichlet to Neumann map on the outer boundary while preserving the combined action. By changing the form in the interior yet again, we could get a Dirichlet to Neumann map where the Neumann operator on the exterior boundary corresponds to the traction boundary condition.

The next theorem provides a uniqueness result for A.

Theorem 3.1. *If $\boldsymbol{w} \in \widetilde{\boldsymbol{H}}_0^1(\Omega_R)$ satisfies*

$$A(\boldsymbol{w}, \boldsymbol{v}) = 0 \quad \textit{for all } \boldsymbol{v} \in \widetilde{\boldsymbol{H}}_0^1(\Omega_R) \tag{3.25}$$

then $\boldsymbol{w} = \mathbf{0}$. We also have that if $\boldsymbol{v} \in \widetilde{\boldsymbol{H}}_0^1(\Omega_R)$ satisfies

$$A(\boldsymbol{w}, \boldsymbol{v}) = 0 \quad \textit{for all } \boldsymbol{w} \in \widetilde{\boldsymbol{H}}_0^1(\Omega_R) \tag{3.26}$$

then $\boldsymbol{v} = \mathbf{0}$.

Proof. If $\boldsymbol{w}$ satisfies (3.25) then, since $A = A_2$, it follows that $A_2(\boldsymbol{w}, \boldsymbol{w}) = 0$. Thus

$$\begin{aligned} 0 = {} & k^2(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}})_{\Omega_{2R}} - (\nabla\times\tilde{\boldsymbol{w}}, \nabla\times\tilde{\boldsymbol{w}})_{\Omega_{2R}} - (1+\gamma)(\nabla\cdot\tilde{\boldsymbol{w}}, \nabla\cdot\tilde{\boldsymbol{w}})_{\Omega_{2R}} \\ & - (\hat{\boldsymbol{x}}\times\nabla\times\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}})_{\Gamma_{2R}} + (1+\gamma)(\nabla\cdot\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}}\cdot\hat{\boldsymbol{x}})_{\Gamma_{2R}}. \end{aligned} \tag{3.27}$$

Now the imaginary part of $A_2(\boldsymbol{w}, \boldsymbol{w})$ vanishes. The first three terms on the right-hand side above are real while the last two are given by the series

$$\sum_{n=0}^{\infty}\sum_{|m|\leqslant n} -\big(\alpha_{n,m}\hat{\boldsymbol{x}}\times\nabla\times(q_n\boldsymbol{V}_{n,m}) + k^2\beta_{n,m}q_n(\hat{\boldsymbol{x}}\times\boldsymbol{V}_{n,m}), \tilde{\boldsymbol{w}}\big)_{\Gamma_{2R}} - k^2(\gamma_{n,m}p_nY_{n,m}, \tilde{\boldsymbol{w}}\cdot\hat{\boldsymbol{x}})_{\Gamma_{2R}}.$$

Applying (3.1) and the identities

$$\boldsymbol{U}_{n,m} = -\hat{\boldsymbol{x}}\times\boldsymbol{V}_{n,m}$$

and

$$\hat{\boldsymbol{x}}\times\nabla\times(q_n\boldsymbol{V}_{n,m}) = -\frac{1}{r}(rq_n)'\boldsymbol{V}_{n,m}$$

give that the above sum reduces to

$$\sum_{n=0}^{\infty}\sum_{|m|\leqslant n}\left[\frac{|\alpha_{n,m}|^2}{r}(rq_n)'\bar{q}_n + k^2\beta_{n,m}q_n\left(\frac{\bar{\gamma}_{n,m}\sqrt{\lambda_n}\bar{p}_n}{r} - \frac{\bar{\beta}_{n,m}}{r}(r\bar{q}_n)'\right) - k^2\gamma_{n,m}p_n\left(\bar{\gamma}_{n,m}\bar{p}_n' - \frac{\bar{\beta}_{n,m}\sqrt{\lambda_n}\bar{q}_n}{r}\right)\right].$$

The above expressions are evaluated at $r = 2R$. The sum of the above terms with coefficients $\beta_{n,m}\bar{\gamma}_{n,m}$ and $\gamma_{n,m}\bar{\beta}_{n,m}$ are real. The imaginary part of the above sum is thus

$$\sum_{n=0}^{\infty}\sum_{|m|\leqslant n}\big[|\alpha_{n,m}|^2\operatorname{Im}(q_n'\bar{q}_n) + k^2|\beta_{n,m}|^2\operatorname{Im}(q_n'\bar{q}_n) + k^2|\gamma_{n,m}|^2\operatorname{Im}(p_n'\bar{p}_n)\big].$$

Using the Wronskian identity $W(h_n^{(1)}(r), h_n^{(2)}(r)) = -2i/r^2$ gives

$$\operatorname{Im}(q_n'\bar{q}_n) = \frac{1}{4kR^2} \quad\text{and}\quad \operatorname{Im}(p_n'\bar{p}_n) = \frac{1}{4k_1R^2}.$$

That the imaginary part of (3.27) is zero immediately implies that $\alpha_{n,m} = \beta_{n,m} = \gamma_{n,m} = 0$ for all n, m, i.e., $\tilde{\boldsymbol{w}}$ vanishes outside of Ω_R. The elasticity equation (3.19) satisfies a unique continuation property which enables us to conclude that $\boldsymbol{w} = \mathbf{0}$ in Ω_R. This verifies the first part of the theorem. The second part is similar. This completes the proof of the theorem. □

Suppose that there is a solution to the time-harmonic elastic wave problem, i.e., a function $\boldsymbol{u} \in \boldsymbol{H}^1_{\text{loc}}(\Omega^c)$ satisfying (2.1) and boundary conditions (2.2), (2.5), (2.8). As observed in Section 2, $\boldsymbol{u}$ is given by the series expansion (2.11) for $r \geqslant R$ and is smooth away from Γ. Accordingly, (2.1) and integration by parts implies that $\boldsymbol{u}$ satisfies

$$A(\boldsymbol{u}, \boldsymbol{\phi}) = 0 \quad \text{for all } \boldsymbol{\phi} \in \widetilde{\boldsymbol{H}}_0^1(\Omega_R).$$

It easily follows from Theorem 3.1 that $\boldsymbol{u}$ is unique on Ω_R and, since it is given by (2.11) outside of Ω_R, it is unique on Ω^c.

The following theorem will be sufficient to guarantee existence.

Theorem 3.2. *There is a positive constant C satisfying*

$$\|\boldsymbol{w}\|_{1,\Omega_R} \leqslant C \sup_{\boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)} \frac{|A(\boldsymbol{w},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{1,\Omega_R}} \quad \text{for all } \boldsymbol{w}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R). \tag{3.28}$$

Proof. We clearly have for $\boldsymbol{w}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)$,

$$\|\tilde{\boldsymbol{w}}\|_{1,\Omega_{2R}} \leqslant \sup_{\boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)} \frac{|\widetilde{A}(\boldsymbol{w},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{1,\Omega_R}}$$

where

$$\widetilde{A}(\boldsymbol{w},\boldsymbol{v}) \equiv (\nabla\tilde{\boldsymbol{w}},\nabla\tilde{\boldsymbol{v}})_{\Omega_{2R}} + \gamma(\nabla\cdot\tilde{\boldsymbol{w}},\nabla\cdot\tilde{\boldsymbol{v}})_{\Omega_{2R}}.$$

Let

$$\mathcal{I}(\boldsymbol{w},\boldsymbol{v}) = k^2(\tilde{\boldsymbol{w}},\tilde{\boldsymbol{v}})_{\Omega_{2R}} + \big(DN_1(\tilde{\boldsymbol{w}}),\tilde{\boldsymbol{v}}\big)_{\Gamma_{2R}}.$$

Then, using (3.22),

$$\begin{aligned} C\|\tilde{\boldsymbol{w}}\|_{1,\Omega_{2R}} &\leqslant \sup_{\boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)} \frac{|A_1(\boldsymbol{w},\boldsymbol{\phi})| + |\mathcal{I}(\boldsymbol{w},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{1,\Omega_R}} \\ &\leqslant \sup_{\boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)} \frac{|A(\boldsymbol{w},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{1,\Omega_R}} + \big[\|\tilde{\boldsymbol{w}}\|_{\Omega_{2R}} + \big\|DN_1(\tilde{\boldsymbol{w}})\big\|_{-1/2,\Gamma_{2R}}\big] \\ &\leqslant \sup_{\boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)} \frac{|A(\boldsymbol{w},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{1,\Omega_R}} + \|\tilde{\boldsymbol{w}}\|_{\boldsymbol{H}^s_{\Omega_{2R}}}, \end{aligned}$$

for any $s\in(1/2,1)$. Now the embedding map from $\boldsymbol{H}^1(\Omega_{2R})$ into $\boldsymbol{H}^s(\Omega_{2R})$ is compact. The inf–sup condition (3.28) now follows from Theorem 3.1 and a lemma by Peetre [11] and Tartar [12] (see also, Theorem 3.2 of [4]). □

The following theorem, which follows easily from Theorems 3.1 and 3.2, is the main result of this paper.

Theorem 3.3. *For any function $\boldsymbol{g}\in\boldsymbol{H}^{1/2}(\Gamma)$, there is a unique solution $\boldsymbol{u}\in\boldsymbol{H}^1_{\mathrm{loc}}(\Omega^c)$ to the elastic wave problem* (2.1) *with boundary conditions* (2.2), (2.5) *and* (2.8)*. Moreover,*

$$\|\boldsymbol{u}\|_{1,\Omega_R} \leqslant C(R)\|\boldsymbol{g}\|_{1/2,\Gamma}. \tag{3.29}$$

Proof. Let $\boldsymbol{u}_g$ be any $\boldsymbol{H}^1(\Omega^c)$ bounded extension of $\boldsymbol{g}$ which vanishes outside of Ω_R. By Theorems 3.1 and 3.2, the generalized Lax–Milgram Lemma [2] (see, also, [4]) implies that there is a unique solution $\boldsymbol{v}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R)$ satisfying

$$A(\boldsymbol{v}+\boldsymbol{u}_g,\boldsymbol{\phi}) = 0 \quad \text{for all } \boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1_0(\Omega_R).$$

It is immediate that $\boldsymbol{u} = \tilde{\boldsymbol{v}} + \boldsymbol{u}_g$ solves the elastic wave problem. Moreover,

$$\|\boldsymbol{u}\|_{1,\Omega_R} \leqslant \|\tilde{\boldsymbol{v}}\|_{1,\Omega_R} + \|\boldsymbol{u}_g\|_{1,\Omega_R} \leqslant C\|\boldsymbol{g}\|_{1/2,\Gamma}$$

so (3.29) follows. Finally, the uniqueness of $\boldsymbol{u}$ was already observed. This finishes the proof of the main result of this paper. □

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